Int. J. Multiphase Flow, Vol. 5, pp. 387–393
Pergamon/Elsevier, 1979. Printed in Great Britain

BRIEF COMMUNICATION

ATTACHED AND FREE EDDIES IN STOKES FLOW

S. H. SMITH
Department of Mathematics, University of Toronto, Toronto, Canada M5S 1A1

(*Received* 12 *February* 1979; *in revised form* 19 *June* 1979)

Abstract—Some simple examples are presented that lead to both attached and free eddies in two-dimensional slow viscous flow. The particular behaviours shown are formed by the combination of elementary solutions of the biharmonic equation.

INTRODUCTION

There has been considerable interest in recent years concerning the question of the existence of separated flows in slow viscous motion. A number of examples have been presented in the literature, and a systematic review has been prepared by O'Neill & Ranger (1979). The first time a separated flow was displayed was in the paper by Dean (1944), who investigated the flow over a particular shaped bump along a wall. Flows with separation had been considered earlier, but the streamlines were not computed and the nature of the behaviour not realised. More recently, O'Neill & Dorrepaal (1979) discovered the existence of free eddies in considering the flow between a pair of circular cylinders, though the circulation between the eddies is extremely weak. Further, the geometry of the obstacle was fundamental to the formation of the eddies in their work; with the free eddies to be displayed here, it is essentially the rotational motion of the body that leads to the detachment of the eddy.

The purpose of this note is to present situations where both attached and free eddies are formed in two-dimensional Stokes flow from the combination of elementary solutions of the biharmonic equation. Specifically, we investigate the linear shear and uniform stream flows past a circular cylinder, combined with fully rotational flows due to a rotation at infinity, plus a separate rotation of the cylinder. With such simple behaviour, the details can be computed quickly and accurately for many different cases, and a general understanding of when such eddies exist can be gained.

There is always a certain artificiality in two-dimensional Stokes flows that try and model behaviour at infinity, and the present study is no exception. A preliminary study of axisymmetric flows involving a sphere has not revealed free eddies of such a simple nature as those given here; there is the example given by Ranger (1971) of the attached eddy when there is a uniform stream past a rotating sphere, and generalizations are possible. With asymmetric flows past a sphere, it does appear that a hyperbolic shear flow coupled with a uniform stream in a rotating fluid can give rise to free eddies; however, with no simple expression for the stream function for these asymmetric flows it is much more difficult to display the eddies. Consequently, the present study is given in its limited form because of its essential simplicity compared to those that have so far been published.

We define the stream function $\psi(r, \theta)$ in polar coordinates, so that the velocity vector $\mathbf{v} = -\text{curl}\,(\psi\hat{\kappa})$; $\hat{\kappa}$ is the unit vector parallel to the cylinder axis. The basic equation is then

$$\nabla_1^4\psi \equiv \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right]^2 \psi = 0, \qquad [1]$$

with boundary conditions $\psi = \psi_r = 0$ or $r = 1$ when the cylinder is at rest, or $\psi = 0$, $\psi_r =$

constant when the cylinder rotates about its axis. The elementary solutions to this equation are

$$\psi_1 = (r^2 - 1 - 2\log r) - \left[r^2 - 2 + \frac{1}{r^2}\right]\cos 2\theta, \qquad [2]$$

$$\psi_2 = \left[r\log r - \frac{r}{2} + \frac{1}{2r}\right]\sin\theta, \qquad [3]$$

$$\psi_3 = r^2 - 1 - 2\log r, \qquad [4]$$

$$\psi_4 = \log r; \qquad [5]$$

ψ_1 represents a linear shear and ψ_2 a uniform stream flow past the circular cylinder, ψ_3 shows rotation at infinity with the cylinder at rest and ψ_4 the flow due to the cylinder itself rotating.

Although ψ_1 satisfies the proper condition $\mathbf{v} \propto y\hat{\imath}$ at infinity for a shear flow, ψ_2 has a logarithmic singularity for the velocity at large distances. However, the work of Proudman & Pearson (1957) indicated that the $r\log r\sin\theta$ behaviour, treated as the outer condition for the inner Stokes solution, is correct for the uniform stream.

Because the governing equation is linear, all combinations

$$\psi = \sum_{i=1}^{4} \lambda_i\psi_i(r, \theta), \qquad [6]$$

for λ_i, also satisfy the equation. We now consider different values for λ_i that display the features of interest.

RESULTS

We formally present five solutions in detail, on the understanding that they show the dominant features. Later, we discuss the range for the constants λ_i for which these features are present, and how they change quantitatively with different values for λ_i.

(a) $\lambda_1 = 1,\ \lambda_2 = 4,\ \lambda_3 = \frac{1}{2},\ \lambda_4 = 0$

We first consider situations where the cylinder is at rest, and the eddy is attached. Figure 1 shows the position of the streamline $\psi = 0$; because there is a symmetry about the y-axis only the eddy in the right half-plane is given. The function ψ is negative within the eddy, and positive for all other points in the fluid. The separation points on the cylinder are where $\theta \simeq 10.5°$ and $\theta \simeq -43.1°$; these are the points where $\psi_{rr} = 0$. The furthest extent of the eddy is at $r \simeq 2.47$, and the minimum value of ψ is -0.199 at $r \simeq 1.86$, $\theta \simeq -16.4°$.

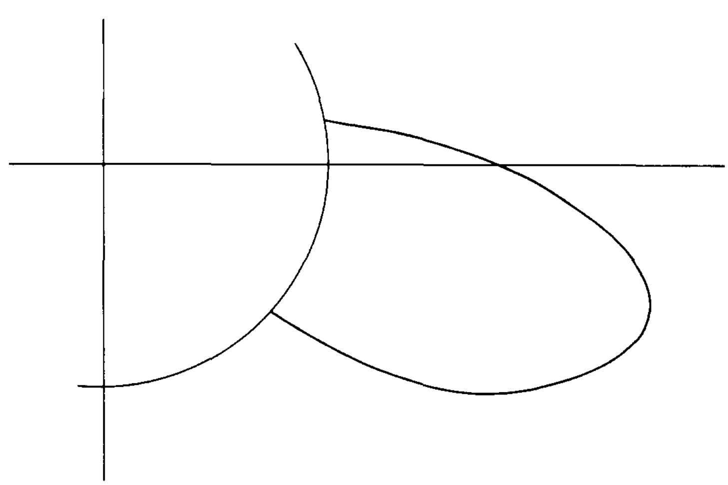

Figure 1. $\lambda_1 = 1,\ \lambda_2 = 4,\ \lambda_3 = \frac{1}{2},\ \lambda_4 = 0.$

(b) $\lambda_1 = 1$, $\lambda_2 = 6$, $\lambda_3 = 1/10$, $\lambda_4 = 0$

Figure 2 shows the eddy, now much larger than in (a) due, it will be seen, to the combined effect of increasing λ_2 and decreasing (positive) λ_3. The separation points are where $\theta \simeq 13.3°$ and $\theta \simeq -78.4°$. It is further seen that ψ_{rrr} is negative in the neighbourhood of the lower separation point, and so the boundary of the eddy curls outward close to the cylinder, although it does not extend beyond $\theta \simeq -79.8°$; this phenomenon is not present near the other separation point. The furthest extent of the eddy is at $r \simeq 17.51$, and the minimum value of ψ is -9.223 at the point $r \simeq 8.90$, $\theta \simeq -17.0°$.

An additional property of interest in the neighbourhood of $\theta = -90°$ can be displayed when the constants λ_1, λ_3 and λ_4 are kept fixed, and λ_2 is slightly increased. For $\lambda = 6.1$, the lower separation point is at $-81.8°$ (with the corresponding point in the left half-plane at $-98.2°$), and the bulge in the eddy boundary extends an extra four degrees to $-85.8°$ (and $-94.2°$); for $\lambda = 6.12$ the lower separation point is at $-82.7°$ (and $-97.3°$), with the bulge now extending to $-88.3°$ (and $-91.7°$). The two eddies are moving closer together, and coalesce when $\lambda_2 \simeq 6.124$ at $r \simeq 1.30$ along $\theta = -90°$. For $6.124 < \lambda < 6.2$ (exactly) there is a distinct eddy with positive circulation embedded within the larger (but now single) eddy with negative circulation. The streamline patterns for $\lambda_2 = 6.12$ and $\lambda_2 = 6.13$ in the neighbourhood of $r = 1$, $\theta = -90°$ are shown together in figure 3. As λ_2 increases within this domain, the eddy where ψ is positive decreases in size until it vanishes when $\lambda_2 = 6.2$.

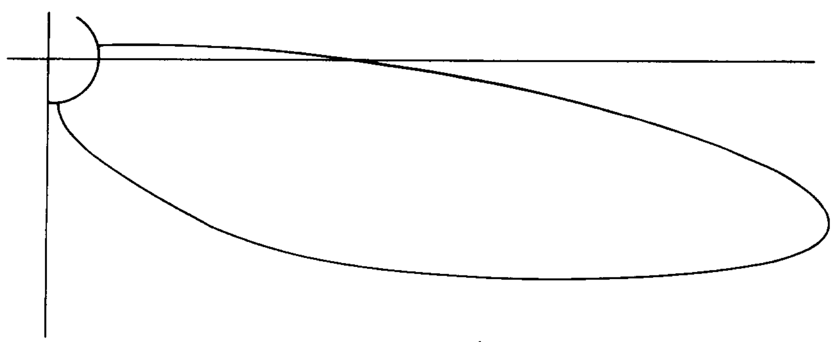

Figure 2. $\lambda_1 = 1$, $\lambda_2 = 6$, $\lambda_3 = \frac{1}{10}$, $\lambda_4 = 0$.

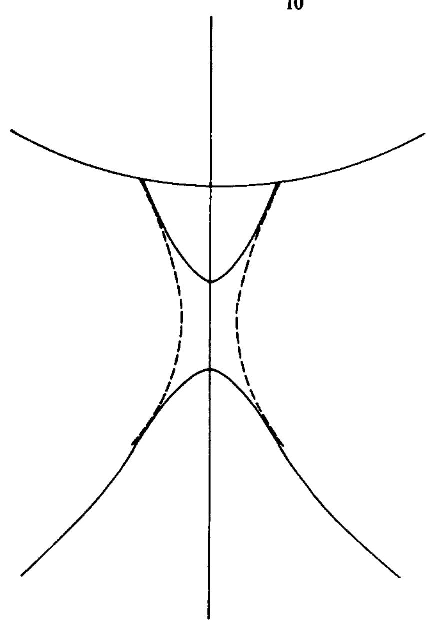

Figure 3. --- $\lambda_1 = 1$, $\lambda_2 = 6.12$, $\lambda_3 = 0.1$, $\lambda_4 = 0$— $\lambda_1 = 1$, $\lambda_2 = 6.13$, $\lambda_3 = 0.1$, $\lambda_4 = 0$.

The separate existence of this small eddy is an interesting feature that does not seem to have been noticed before. Mathematically, it is a consequence of the fact that the ratio

$$\left[r^2-2+\frac{1}{r^2}\right]\Big/\left[r\log r-\frac{r}{2}+\frac{1}{2r}\right]$$

initially decreases for $1 \leq r < 2.13$, before then increasing once $r > 2.13$; each of the other ratios that can be formed from the fundamental solutions [2]–[5] are monotonic for all $r \geq 1$.

Some general statements can now be added to indicate the changes in the behaviour as the constants λ_2 and λ_3 change, (only two of the three are independent, and $\lambda_1 = 1$ is maintained as the scaling factor). In the following, we do not put any significance on properties describing situations far from the cylinder on the basis that a Stokes flow is an "inner" solution in the sense of Proudman & Pearson, and so must be seen mainly as a local description.

Separation takes place when $\lambda_2^2 > 64(\lambda_3 - 1)$, the bulge appears at the lower separation point when $64(\lambda_3+1)-\lambda_2^2 > \lambda_2\{\lambda_2^2+64(1-\lambda_3)\}^{1/2}$, and the two halves of the eddy are fully merged when $\lambda_2 > 6+2\lambda_3$; these domains in the λ_2, λ_3-plane are represented by the horizontally, vertically and cross hatched regions respectively in figure 4. The position of the eddy is essentially governed by the value of λ_2, with the separation points satisfying $16\sin\theta = -\lambda_2 \pm \{\lambda_2^2+64(1-\lambda_3)\}^{1/2}$; as λ_3 increases, for fixed λ_2, the eddy vanishes where $\theta = -\arcsin(\lambda_2/16)$. For any fixed λ_3, the size and circulation of the eddy increase with λ_2.

(c) $\lambda_1 = 1$, $\lambda_2 = 4$, $\lambda_3 = \frac{1}{2}$, $\lambda_4 = \frac{1}{5}$

As soon as there is a slight rotation by the cylinder added to example (a), the eddy is swept off the cylinder and becomes free. For very small λ_4, figure 1 would still be appropriate except for points in the region close to the surface of the cylinder. When the particular (larger) value $\lambda_4 = \frac{1}{5}$ is adopted, the free eddy takes the form given in figure 5. There is a stagnation point S at $r \simeq 1.0791$ and $\theta \simeq 14.84°$; the streamline sketched is that which passes through S, and is given by $\psi \simeq 0.00726$. The maximum extent of the eddy is $r \simeq 2.16$, and it is contained between the radius vectors $\theta \simeq -3.5°$ and $-28.6°$. The minimum value for ψ is -0.0814, at $r \simeq 1.74$, $\theta \simeq -16.4°$. The streamline $\psi = 0.00726$ is distant only 0.0182 from the cylinder at its closest point.

To give an idea of the role of small (positive) rotation by the cylinder, the stagnation point equivalent to S is positioned at $r = 1+\frac{1}{3}\epsilon$, $\theta = -\arcsin(\frac{1}{4})$ when $\lambda_4 = \epsilon \ll 1$.

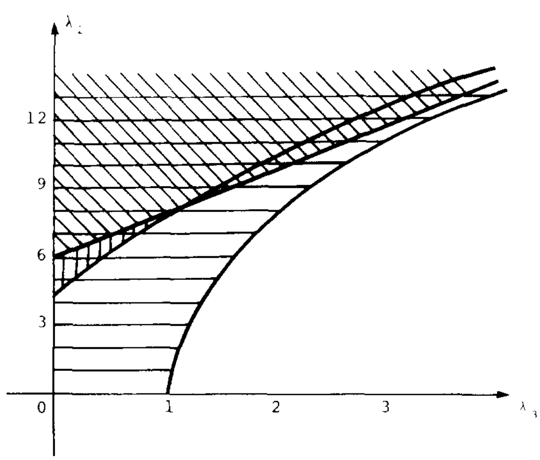

Figure 4. Domains for separation (examples A and B).

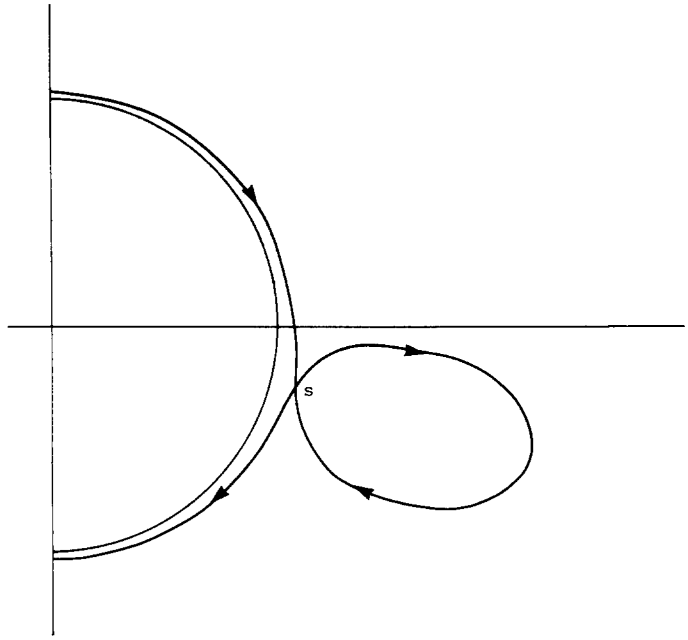

Figure 5. $\lambda_1 = 1$, $\lambda_2 = 4$, $\lambda_3 = \frac{1}{2}$, $\lambda_4 = \frac{1}{5}$.

(d) $\lambda_1 = 1$, $\lambda_2 = 4$, $\lambda_3 = \frac{1}{2}$, $\lambda_4 = -\frac{1}{5}$

When the rotation is in the opposite direction, the stagnation points lie on the line of symmetry $\theta = \pm 90°$. In the particular case under consideration they are positioned at $S_1(r \simeq 1.0091, \theta = 90°)$ and $S_2(r \simeq 1.0335, \theta = -90°)$. The boundary of the free eddy is given in figure 6; it is the streamline where $\psi \simeq -0.00331$ and passes through S_2 given by $r \simeq 1.015$ at $\theta = 0$. A typical distance of the streamline from the cylinder is given by $r \simeq 1.015$ at $\theta = 0$. The maximum extent of the eddy is $r \simeq 2.72$, and it lies below the arc $\theta \simeq 29.5°$. The minimum value for ψ is -0.329 at $r \simeq 1.96$, $\theta \simeq -16.4°$.

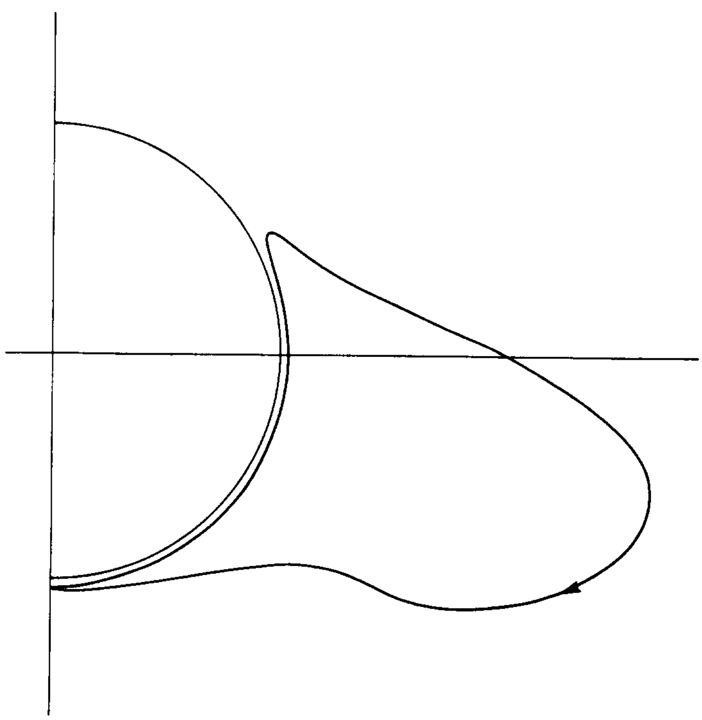

Figure 6. $\lambda_1 = 1$, $\lambda_2 = 4$, $\lambda_3 = \frac{1}{2}$, $\lambda_4 = -\frac{1}{5}$.

The streamline $\psi \simeq -0.00091$ passes through S_1, and traces a path both inside and outside that given in figure 6, but this cannot easily be represented in the diagram. Geometrically, there is a saddle point at both S_1 and S_2.

(e) $\lambda_1 = 1$, $\lambda_2 = 6$, $\lambda_3 = \frac{1}{10}$, $\lambda_4 = \frac{11}{5}$

The eddy boundary is shown in figure 7. The stagnation point is at $r \simeq 1.509$, $\theta \simeq -24.46°$, and the streamline through this point is $\psi \simeq 0.443$. It is bounded between the arcs $\theta \simeq -5.7°$ and $-41.7°$; the maximum extent is $r \simeq 14.52$. The minimum value for ψ is -4.426 at $r \simeq 8.42$ and $\theta \simeq -17.5°$.

It is also of interest to consider the effect of a small amount of circulation on the example presented in figure 3. It is difficult to present these diagramatically because of the closeness of the relevant streamlines, and so the following description must suffice. First we take $\lambda_1 = 1$, $\lambda_2 = 6.13$, $\lambda_3 = 0.1$, $\lambda_4 = -0.001$; any value $\lambda_4 \leq -0.00234$ would remove this small eddy altogether. The free eddy with positive circulation is bounded by the streamline $\psi \simeq -0.0000039$, which passes through the stagnation point $r \simeq 1.0071$, $\theta = -90°$, and extends no further than $r \simeq 1.16$ from the cylinder. The eddy is also bounded by the arcs $|\theta + 90°| \simeq 4.1°$; the shape is not dissimilar from that given in figure 3 when detached from the cylinder.

When λ_4 is changed to $+0.001$ however, the shape of the free eddy does alter considerably from figure 3. The dividing streamline $\psi \simeq 0.000000084$ now shows a thin lens-like region with the cusps being stagnation points at $r \simeq 1.00017$, $\theta \simeq -22.5°$ and $-157.5°$. The larger of the two arcs cuts $\theta = -90°$ at $r \simeq 1.245$, which represents the furthest distance of the boundary from the cylinder, and the smaller cuts $\theta = -90°$ at $r \simeq = 1.000084$. The maximum value of ψ within the free eddy is 0.00035 at $r \simeq 1.127$, $\theta = -90°$.

General statements equivalent to those given earlier for the detached eddy are not as simply stated when there are three independent parameters; however, when we generalize (e) by maintaining (for arithmetic convenience alone) $\lambda_4 = 2(\lambda_3 + 1)$, figure 8 shows the range of the parameters λ_2, λ_3 for which the free eddy exists.

To conclude, it can be said that further cases have been considered. When λ_2 changes sign, this serves only to reflect the streamline curves in the x-axis. When $\lambda_3 < 0$ and $\lambda_4 > 0$, no case was found where a free eddy exists; changing signs for both λ_3 and λ_4 just reverses the direction of rotational flow. When $-2 < \lambda_3/\lambda_1 < 0$ in particular, the streamline $\psi = 0$ extends to infinity.

DISCUSSION

The main purpose of the present paper is to exhibit these very simple examples of separation in Stokes flows that lead to both attached and free eddies. It is hoped that their very

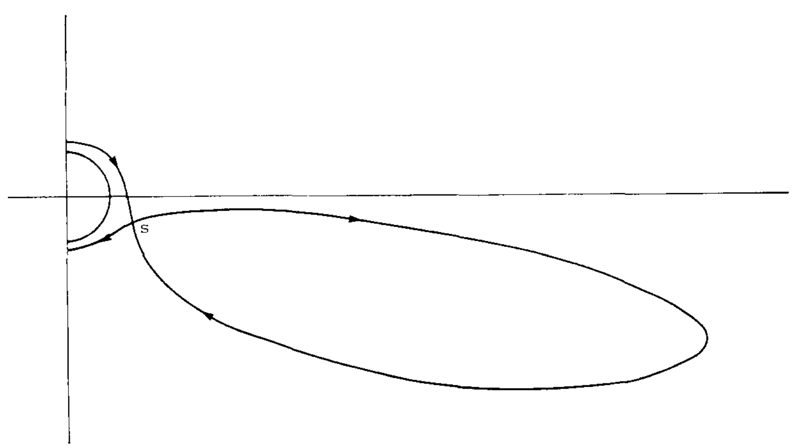

Figure 7. $\lambda_1 = 1$, $\lambda_2 = 6$, $\lambda_3 = \frac{1}{10}$, $\lambda_4 = \frac{11}{5}$.

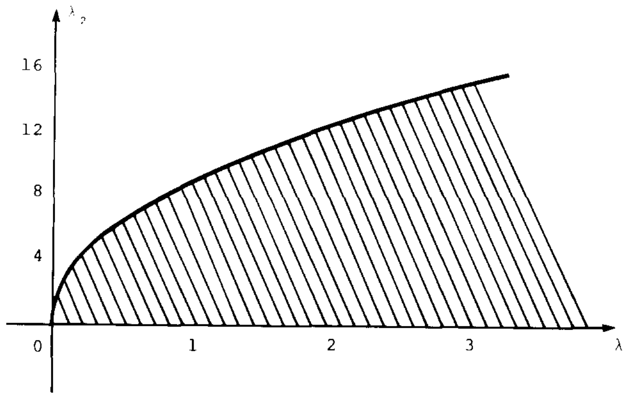

Figure 8. Domain for free eddy with $\lambda_4 = 2(\lambda_3 + 1)$ (example D).

simplicity, as distinct from the detailed analysis of the previous results, will make the concepts more accessible, for it can reasonably be said that the possibility of separation should always be considered. The reason separation has not been so seriously considered until recently is in part due to the fact that in only a few specific cases have the streamlines been developed. It is hard enough to solve the biharmonic equation when two boundaries are present, and harder still to determine clearly the position of the streamlines.

However, it is clear that the combination of ψ_1 and ψ_2, representing a displaced shear flow (i.e. off the centre line of the cylinder), is the common local flow in the neighbourhood of a body. Further, the main role of including small quantities of ψ_3 is essentially to close the streamline at infinity. Therefore, it can be inferred from the general discussion after example (b) that separation in linear shear flows past a cylindrical body is indeed a common phenomenon. And once rotation (for a circular cylinder), or any general motion of a body in the plane normal to its axis is included, a previously attached separation bubble must become detached.

As an example, Davis & O'Neill (1977) have recently shown the existence of separation in the linear shear flow past a circular cylinder close to an infinite flat plate. It is now clear, when the cylinder rotates slowly enough, or the plate itself moves slowly in its own plane, that the eddies they found attached to the bodies will become free. Other examples can readily be developed.

It can now be argued that the phenomenon of separation will be understood from studies of low just as much as high Reynolds number flows, and further specific examples of the former need to be developed.

REFERENCES

Davis, A. M. J. & O'Neill, M. E. 1977 Separation in a slow linear shear flow past a cylinder and a plane. *J. Fluid Mech.* **82**, 551–564.

Dean, W. R. 1944 On the shearing motion of fluid past a projection. *Proc. Camb. Phil Soc.* **40**, 18–36.

O'Neill, M. E. & Dorrepaal, J. M. 1979 The existence of free eddies in a streaming Stokes flow. *Q. J. Mech. and Appl. Math.* in press.

O'Neill, M. E. & Ranger, K. M. 1979 Laminar particle fluid interaction. In *Handbook of Multiphase Flow* (Edited by Hetsroni, G.). Hemisphere, Washington.

Proudman, I. & Pearson, J. R. A. 1957 Expansions at small Reynolds numbers for the flow past a sphere and a circular cylinder. *J. Fluid Mech.* **2**, 237–262.

Ranger, K. B. 1971 Slow viscous flow past a rotating sphere. *Proc. Camb. Phil. Soc.* **69**, 333–336.